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ON THE MOTION OF DYNAMICALLY CONTROLLED SYSTEMS WITH VARIABLE MASSES*

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Equations of motion are derived for a mechanical system with variable masses and superimposed constraints, whose responses are reactive forces. A theory is developed for the solution of an optimal motion control problem for this kind of system when reactive forces are taken as controls. As an example a problem on contact in minimal time is solved by the method of parallel approach of some target and a system of point of variable mass.

1. We analyze a mechanical system of n mass points P_k $(k = 1, \ldots, n)$ whose positions in an absolute frame of reference are determined by their Cartesian coordinates x_ν $(\nu = 1, \ldots, 3n)$. Let prescribed forces $F_{k}(X_\nu)$ act on points P_k and let their motion be subject to compatible and independent constraints

$$f_\alpha(x_\nu, x_\nu^{\cdot}, t) = 0 \quad (\alpha = 1, \ldots, a) \tag{1.1}$$

among which a_1 constraints are geometric. The virtual displacements, admitting of superimposed constraints, are determined by a independent relations /1/

$$\sum_{\nu=1}^{3n} \frac{\partial f_\alpha}{\partial x_\nu^{\cdot}} \delta x_\nu = 0 \quad (\alpha = 1, \ldots, a)$$

while the system's configuration is described by $h = 3n - a_1$ independent Lagrangian coordinates $q_1, \ldots, q_h$. Because of the $a_2 = h - a_1$ nonholonomic constraints in (1.1) the variations of the latter coordinates are connected by a_2 conditions

$$\sum_{j=1}^{h} \sum_{\nu=1}^{3n} \frac{\partial f_\alpha}{\partial x_\nu^{\cdot}} \frac{\partial x_\nu^{\cdot}}{\partial q_j^{\cdot}} \delta q_j = 0 \quad (\alpha = a_1 + 1, \ldots, a)$$

which permit us to express a_2 independent variations as linear homogeneous functions of l independent quantities δq_i $(i = 1, \ldots, l)$. The variations of the Cartesian coordinates of the system's points take the form

$$\delta x_\nu = \sum_{i=1}^{l} c_{i\nu} \delta q_i \quad (\nu = 1, \ldots, 3n)$$

Here $c_{i\nu}(q_j, q_j^{\cdot}, t)$ are known functions of the variables indicated, while the quantities δq_i are arbitrary.

2. As is well known /2/, the constraints imposed on the system depend upon the physical nature of the mechanisms effecting them, in view of which the characteristic of the constraints is introduced by an axiom expressing the actually existing empirical relations. We assume that the mechanical system being analyzed is a system of mass points with variable masses, while the constraints are imposed by reactive forces that go into action automatically and are automatically regulated. In other words, let the responses to the constraints being examined be reactive forces R_k (R_ν) going into action automatically and such that the accelerations of the points at any instant and for any positions and velocities consistent with the constraints form a system of feasible accelerations, i.e., do not contradict the conditions

$$\sum_{\nu=1}^{3n} \frac{\partial f_\alpha}{\partial x_\nu^{\cdot}} x_\nu^{\cdot\cdot} + e_\alpha(x_\nu, x_\nu^{\cdot}, t) = 0 \quad (\alpha = 1, \ldots, a) \tag{2.1}$$

The equality

$$\sum_{\nu=1}^{3n} R_\nu \delta x_\nu = 0$$

valid for any virtual displacements, serves as an axiom of ideal constraints. The necessary and sufficient condition for this is the fulfillment of the conditions /2/

$$R_\nu = \sum_{\alpha=1}^{a} \lambda_\alpha \frac{\partial f_\alpha}{\partial x_\nu^{\cdot}} \quad (\nu = 1, \ldots, 3n)$$

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Let the sum of the elementary works of the reactive forces at every virtual displacement equal $b \neq 0$. Then there exists an infinite set of reactive forces R_k' (R_ν') possessing the property that $R_1'\delta x_1 + R_2'\delta x_2 + \ldots + R_{3n}'\delta x_{3n} = b$ for every virtual displacement. The fulfillment of the equalities

$$R_\nu' = R_\nu + \sum_{\alpha=1}^{a} \lambda_\alpha \frac{\partial f_\alpha}{\partial x_\nu} \quad (\nu = 1, \ldots, 3n)$$

is necessary and sufficient for this. Among the systems of R_k' there exists one and only one system of reactive forces Φ_k (Φ_ν) such that the vectors $\Phi_k \delta t$ define a certain virtual displacement $\delta x_\nu = \Phi_\nu \delta t$ ($\nu = 1, \ldots, 3n$). Indeed, let

$$\sum_{\nu=1}^{3n} \Phi_\nu \delta x_\nu = \sum_{\nu=1}^{3n} R_\nu \delta x_\nu$$

on every virtual displacement. Hence, by virtue of the independence of the quantities δq_i we have the l equations

$$\sum_{\nu=1}^{3n} (\Phi_\nu - R_\nu) c_{\nu i} = 0 \quad (i = 1, \ldots, l)$$

which jointly with the a equations

$$\sum_{\nu=1}^{3n} \frac{\partial f_\alpha}{\partial x_\nu} \Phi_\nu = 0 \quad (\alpha = 1, \ldots, a)$$

form a system of $a + l = 3n$ equations for the determination of the $3n$ unknowns Φ_ν ($\nu = 1, \ldots, 3n$). The determinant of this system is nonzero because otherwise there would exist a system of forces $\Phi_k \not\equiv 0$ for which

$$\sum_{k=1}^{n} \Phi_k \delta r_k = \delta t \sum_{k=1}^{n} \Phi_k^2 = 0$$

is not possible. Consequently, there exists one and only one system of variables Φ_k ($k = 1, \ldots, l$), while the reactive force R_k can be uniquely decomposed into two components N_k and Φ_k such that $N_1\delta x_1 + N_2\delta x_2 + \ldots + N_{3n}\delta x_{3n} = 0$ on every virtual displacement, while the vectors $\Phi_k\delta t$ are found among the virtual displacements. Here

$$N_\nu = \sum_{\alpha=1}^{a} \lambda_\alpha \frac{\partial f_\alpha}{\partial x_\nu}, \quad \Phi_\nu = \sum_{i=1}^{h} u_i c_{i\nu} \quad (\nu = 1, \ldots, 3n)$$

the coefficients λ_α and u_i are the same for all points of the system. The quantity N_k is called a reactive constraint force, while Φ_k is called a reactive thrust force.

If at the instant t being examined we know the positions, the velocities, the laws of variation of the masses of the system's points, and the acting forces, then the constraint forces are determined uniquely and are one and the same independently of whether or not the system possesses a thrust. Indeed, the equations of motions of the points can be written as

$$m x_\nu'' = X_\nu + \sum_{\alpha=1}^{a} \lambda_\alpha \frac{\partial f_\alpha}{\partial x_\nu} + \sum_{i=1}^{h} u_i c_{i\nu} \quad (\nu = 1, \ldots, 3n) \tag{2.2}$$

Substituting from here the quantities x_ν'' into Eq. (2.1), we obtain a system of a linear equations in the a variables λ_α. Consequently, for a system not possessing thrust a knowledge of the external forces under specified initial conditions and point mass variation laws is sufficient for the determination of the motion and of the constraint forces. If the laws of variation of the thrust forces are known, then for the description of the system's motion we have the $3n$ equations (2.2) to which we should add on the a constraint equations and the l additional relations obtained from the thrust law,

3. Let nonholonomic constraints be absent in relations (1.1). Then the variations of the Cartesian coordinates are expressed by the relations

$$\delta x_\nu = \sum_{j=1}^{h} \frac{\partial x_\nu}{\partial q_j} \delta q_j \quad (\nu = 1, \ldots, 3n)$$

We multiply each of the equations of system (2.2) by δx_ν and we add. After transformations we have

$$\sum_{j=1}^{h} \left[\sum_{\nu=1}^{3n} m_\nu \left(\frac{d}{dt} \frac{\partial}{\partial q_j'} \frac{x_\nu'^2}{2} - \frac{\partial}{\partial q_j} \frac{x_\nu'^2}{2} \right) \right] \delta q_j = \sum_{j=1}^{h} Q_j \delta q_j + \sum_{j=1}^{h} \left(\sum_{i=1}^{h} u_i \sum_{\nu=1}^{3n} \frac{\partial x_\nu}{\partial q_i} \frac{\partial x_\nu}{\partial q_j} \right) \delta q_j \tag{3.1}$$

We see that

$$\sum_{\nu=1}^{3n} \frac{\partial x_\nu}{\partial q_i} \frac{\partial x_\nu}{\partial q_j} = \frac{\partial}{\partial q_i^{\cdot}}\left(\frac{\partial \tau}{\partial q_j^{\cdot}} - \frac{\partial s}{\partial q_j}\right) + \frac{\partial}{\partial q_i}\frac{\partial \pi}{\partial q_j}, \qquad \tau = \frac{1}{2}\sum_{\nu=1}^{3n} x_\nu^{\cdot 2}, \quad s = \sum_{\nu=1}^{3n} x_\nu x_\nu^{\cdot}, \quad \pi = \frac{1}{2}\sum_{\nu=1}^{3n} x_\nu^2$$

We introduce the notation

$$\frac{d'}{dt}\frac{\partial' T}{\partial q_j^{\cdot}} = \sum_{\nu=1}^{3n} m_\nu \frac{d}{dt}\frac{\partial}{\partial q_j^{\cdot}}\frac{x_\nu^{\cdot 2}}{2}, \quad \frac{\partial' T}{\partial q_j} = \sum_{\nu=1}^{3n} m_\nu \frac{\partial}{\partial q_j}\frac{x_\nu^{\cdot 2}}{2}$$

and, because the quantities δq_j are independent, from (3.1) be obtain the equations of motion of the system in the form of Lagrange equations of the second kind

$$\frac{d'}{dt}\frac{\partial' T}{\partial q_j^{\cdot}} - \frac{\partial' T}{\partial q_j} = Q_j + \sum_{i=1}^{h} u_i\left[\frac{\partial}{\partial q_i^{\cdot}}\left(\frac{\partial \tau}{\partial q_j^{\cdot}} - \frac{\partial s}{\partial q_j}\right) + \frac{\partial}{\partial q_i}\frac{\partial \pi}{\partial q_j}\right] \quad (j = 1, \ldots h) \tag{3.2}$$

$$T = \frac{1}{2}\sum_{i,j=1}^{h} A_{ij}(q_j, q_j^{\cdot}, t) q_i^{\cdot} q_j^{\cdot} + \sum_{i=1}^{h} A_i(q_j, q_j^{\cdot}, t) q_i^{\cdot} + \frac{1}{2} T_0(q_j, q_j^{\cdot}, t) \quad (m_\nu = m_\nu(q_j, q_j^{\cdot}, t))$$

The quantities

$$p_j = \sum m_\nu \frac{\partial}{\partial q_j^{\cdot}}\frac{x_\nu^{\cdot 2}}{2} = \frac{\partial' T}{\partial q_j^{\cdot}} \quad (j = 1, \ldots, h) \tag{3.3}$$

are called generalized momenta. Setting $(\delta q_j)^{\cdot} = \delta q_j^{\cdot}$, we represent (3.1) as a central Lagrange equation

$$\frac{d'}{dt}\sum_{j=1}^{h} p_j \delta q_j = \delta T + \sum_{j=1}^{h} Q_j \delta q_j + \sum_{j=1}^{h}\left\{\sum_{i=1}^{h} u_i\left[\frac{\partial}{\partial q_j^{\cdot}}\left(\frac{\partial \tau}{\partial q_j^{\cdot}} - \frac{\partial s}{\partial q_j}\right) + \frac{\partial}{\partial q_i}\frac{\partial \pi}{\partial q_j}\right]\right\}\delta q_j$$

The kinetic energy's Hessian, computed under the assumption of constancy of the system's mass, cannot be zero; therefore, (3.3) is solvable relative to the generalized velocities $q_j^{\cdot}$, while the generating function of the inverse transformation is determined in the form

$$K = \sum_{j=1}^{h} p_j q_j^{\cdot} - T(q_j, q_j^{\cdot}, t)$$

Function K together with the central Lagrange equations enables us to give the system's equations of motion the canonic form

$$\frac{d' p_j}{dt} = -\frac{\partial' K}{\partial q_j} + Q_j + \sum_{i=1}^{h} u_i\left[\frac{\partial}{\partial q_i^{\cdot}}\left(\frac{\partial \tau}{\partial q_j^{\cdot}} - \frac{\partial s}{\partial q_j}\right) + \frac{\partial}{\partial q_i}\frac{\partial \pi}{\partial q_j}\right], \qquad \frac{d' q_j}{dt} = \frac{\partial' K}{\partial p_j} \quad (j = 1, \ldots, h) \tag{3.4}$$

In the right hand sides of the first h equations the generalized forces Q_j and the quantities within the brackets are assumed to be expressed in terms of the generalized coordinates and momenta. The equations obtained define the relative motion of the mapping point describing the system's state, in the deformed $2h$-dimensional phase space. If the quantities u_i, the mass variation laws, and the system's initial state $(q_1^\circ, \ldots, q_h^\circ, p_1^\circ, \ldots, p_h^\circ)$ are specified, then the system's behavior - the trajectories in phase space - uniquely defined. However, if the quantities u_i are not specified in advance, then the resultant indeterminacy proves to be useful when considering motion modes that are optimal in some sense or other. Indeed, a mechanical system with a known mass variation law, being investigated, can in such case be treated as an object of automatic regulation, described by a system of $2h$ first-order differential equations (3.4) and having h regulating organs whose positions are determined by the h parameters u_i. Then the main problem is to choose the control $u = \{u_1, \ldots, u_h\}$ under which the system's behavior becomes optimal in some predetermined sense. Obviously, in the general case it is necessary to treat certain characteristics of the mass variation of the system's points as controlling parameters too.

4. As an example we consider one particular problem of exterior ballistics. Let a target A, whose motion is prescribed at any instant t $(t_0 \leqslant t \leqslant t_1)$, be pursued by a system of n controlled points with variable masses. It is assumed that each pursuing point is guided by the parallel approach method /3/ and that control by automatically controllable reactive forces ensures that all n pursuing points with arbitrary contact velocities hit on the target simultaneously (at instant t_1). It is required to determine the motion of the pursuing system under prescribed initial data, if it is known that the mass of each pursuing point varies by the law

$$m_\nu = m_{\nu_0} \exp\left[\int_{t_0}^{t} f(t)\,dt\right] = m_{\nu_0}\gamma(t) \quad (\nu = 1, \ldots, 3n)$$

while the approach must take place in minimal time. We assume that target A and the points pursuing it form a similarly changing system with variable mass /4/. Then the simultaneous hitting of the target is ensured and the question is reduced to the study of the translational motion of a similarly changing body under a prescribed motion of one of its points. In this case the body possesses one degree of freedom and its position is determined by one generalized coordinate μ called the radiating compression function. It is seen that for the mass variation law adopted the principal central inertia axes in the body remain unchanged. Consequently, the body's kinetic energy, referred to these axes and written with due regard to the known law of motion of the target, is determined by the relation

$$2T = [M_0 (F(t) + \mu^{\cdot} a) - 2M_0 \mu^{\cdot} Q(t) + \mu^{\cdot 2} \Pi_0] \gamma(t)$$

where $F(t)$ and $Q(t)$ are known time functions stipulated by the target's motion, a, M_0, Π_0 are constant depending on the initial data. The generating function of the inverse transformation is

$$K = p^2 [2b\gamma(t)]^{-1} + M_0 b^{-1} Q(t) p + 2^{-1} M_0 [M_0 b^{-1} Q^2(t) - F(t)] \gamma(t)$$

$$b = M_0 a + \Pi_0 = \text{const}, \quad p = [\mu^{\cdot} b - M_0 Q(t)] \gamma(t)$$

In addition, the equalities

$$2\tau = \mu^{\cdot 2} k + 2\mu L(t) + F(t), \quad s = \mu\mu^{\cdot} k + \mu^{\cdot} M(t) + \mu L(t) + v(t)$$

$$2\pi = \mu^2 k + 2\mu M(t) + w(t), \quad k = \text{const}$$

hold. Here L, M, v, w are known time functions.

Let the external forces be absent and let the target's motion be uniform and rectilinear ($Q(t) = d = \text{const}$). Then the canonic equation system (3.4) is written as

$$\mu^{\cdot} = p b^{-1} \gamma^{-1}(t) + M_0 d b^{-1}, \quad p^{\cdot} = ku + f(t) p \tag{4.1}$$

For the case given the components of the reactive thrust forces are determined by

$$\Phi_\nu = u \partial x_\nu / \partial \mu \quad (\nu = 1, \ldots, 3n)$$

We bound the absolute value of each thrust force by some limit, the same for each point

$$0 \leqslant |\Phi_k| \leqslant |\Phi_{\max}| \quad (k = 1, \ldots, n)$$

However, for an individual point

$$|\Phi_k| = \Big[u^2 \sum_{i=1}^{3} (\partial x_k^i / \partial \mu)^2\Big]^{1/2} = |u l_k^\circ| \quad (k = 1, \ldots, n)$$

where l_k° is the initial distance of the k-th point of the body from the initial position of the center of mass. Consequently, the greatest thrust force must be developed for the point with the largest value of l_k°. Hence we obtain

$$0 \leqslant |u| \leqslant |\Phi_{\max} / l_{\max}^\circ| \tag{4.2}$$

imposing a constraint on the regulation parameter u and determining its admissible values.

Let us find how the quantity u must vary under the specified law $f(t)$ in order that function $\mu(t)$ vanish in minimal time with an arbitrary value of $\mu^{\cdot}(t_1)$, and, consequently, $p(t)$. In other words, we consider the problem of the most rapid hitting of an object guided by Eqs. (4.1) and (4.2), into some point of a manifold defined by the equation $\mu = 0$, from some initial position (μ_0, p_0) on the phase plane. We find the required optimal value of u from the condition of the maximum over u of the function /5/

$$H = \psi_1 [p b^{-1} \gamma^{-1}(t) + M_0 d b^{-1}] + \psi_2 [ku + f(t) p]$$

As is seen, this function has a maximum with respect to u for

$$u = |\Phi_{\max} / l_{\max}^\circ| \operatorname{sign} \psi_2$$

The auxiliary variables ψ_1 and ψ_2 are determined by the relations

$$\psi_1^{\cdot} = -\partial H / \partial \mu = 0, \quad \psi_2^{\cdot} = -\partial H / \partial p = -\psi_1 b^{-1} \gamma^{-1}(t) - \psi_2 f(t)$$

Hence $\psi_1 = c_1$ and $\psi_2 = (c_2 - c_1 b^{-1} t) \gamma^{-1}(t)$, where c_1 and c_2 are constants. We note that because $p(t_1)$ is arbitrary we can analyze a time-optimal problem with a moving right endpoint. The vector $\theta = \{\theta_1, \theta_2\}$, tangent to manifold $\mu = 0$, has the form $\theta = \{0, \theta_2\}$, where $\theta_2 \neq 0$. Consequently, the transversality condition at the trajectory's right end can be written as $0 \cdot \psi_1(t_1) + \theta_2 \psi_2(t_1) = 0$. Hence

$$\psi_2(t_1) = 0, \quad c_2 = c_1 b^{-1} t_1, \quad \psi_2 = c_1 (t_1 - t) / (b\gamma(t))$$

As we see, function $\psi_2(t)$ preserves sign for $t_0 \leqslant t \leqslant t_1$. Consequently, each optimal value of control parameter u is constant, nonzero and equal to $\pm |\Phi_{\max} / l_{\max}^\circ|$ depending on the sign of c_1. In particular, for $f(t) = -\beta (1 - \beta t)^{-1}$ ($\beta = \text{const}$, $1 - \beta t > 0$) the expressions for momentum p and coordinate μ are

$$p = p_0 (1 - \beta t) \pm k\beta^{-1} u (1 - \beta t) \ln (1 - \beta t)$$
$$\mu = b^{-1} (p_0 - M_0 d) t \pm k\beta^{-2} u (1 - \beta t) \ln (1 - \beta t) \pm k\beta^{-1} ut + 1$$

The upper signs correspond to positive values of c_1, the lower to negative; when $p_0 > 0$ ($p_0 < 0$) we should set $c_1 > 0$ ($c_1 < 0$) since the condition that the maximum of function H be positive must be fulfilled at the final instant t_1.

REFERENCES

1. CHETAEV, N. G., Papers on Analytical Mechanics. On the Gauss principle. Moscow, Izd. Akad. Nauk SSSR, 1962.
2. RUMIANTSEV. V. V., On systems with friction. PMM Vol.25, No.6, 1961.
3. MERRILL, G. (Ed.), Principles of Guided Missile Design. Vol.1: Locke, A. S., Guidance. Princeton, NJ. D. Van Nostrand Co., Inc., 1955.
4. IAKOVLEV, V. F., On the motion of certain mechanical systems with variable masses. Izv. Akad. Nauk SSSR, Otdel. Tekhn. Nauk, Mekh. i Mashinostr., No.5, 1963.
5. PONTRIAGIN, L. S., BOLTIANSKII, V. G., GAMKRELIDZE, R. V. and MISHCHENKO, E. F., Mathematical Theory of Optimal Processes. English translation, Pergamon Press, Book No. 10176, 1964.

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